Title: Arithmetic Chains

Brief Overview:

Students will complete a chain of numbers based on given sums to arrive at a final “target” number. The goal is for them to find a way of predicting the target based on the starting number without going through the entire process. They will discover this inductively and use algebra to prove that their prediction method is correct. They will generalize this to chains using other sums and containing more links.

Links to NCTM 2000 Standards:

- **Mathematics as Problem Solving**
  Students will solve the problem of finding a quick way of predicting the target number without having to go through the entire process of calculating the chain of numbers every time.

- **Mathematics as Reasoning and Proof**
  Students will use inductive reasoning to conjecture a way of predicting the target number from the start number for various sums and for various numbers of links. Then they will use deductive reasoning to prove algebraically that their method is correct.

- **Mathematics as Communication**
  Students will discuss their findings with each other and write their proofs using algebraic notation.

- **Mathematics as Connections**
  Students will discover the transition from arithmetic to algebra that this unit provides.

- **Mathematics as Representation**
  Students will represent the patterns they discover using algebraic notation.

- **Patterns, Functions, and Algebra**
  Students will be informally studying the concept of function, although function terminology and notation will not be used. They will find patterns in the target numbers as a function of the start numbers. They will use algebra to express their discoveries and prove them.

Grade/Level:

Grades 8 - 9 (and gifted 7th)

Duration/Length:

2 - 3 days, depending on students’ prior knowledge

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Translating English phrases to algebraic expressions
- Simplifying algebraic expressions

Student Outcomes:

Students will:

- search for patterns and represent their conjectures algebraically.
- prove their conjectures using algebra.

Materials/Resources/Printed Materials:

- Worksheets 1 & 2
- Spreadsheet software (optional)

Development/Procedures:

- Pass out Worksheet 1, and explain how the students are to fill in each arithmetic chain. Moving left to right, each square is filled in with the number which when added to the number to its left, gives the sum below. For example, starting with 3, the number 4 is placed in the first empty square to give a sum of 7. Similarly, the next square is filled in with 8, and the last square (the target) is filled in with 7. Have students complete the next two chains in part A, using any number they wish for the starting number in the third chain. (The second one yields 5.)

- Ask them if they see any pattern in the results. How are the start and target numbers related? They should notice that their sum is 10. Have them choose any number to start the fourth chain and predict what they will get for the target. Then have them work through it step-by-step to confirm their prediction. Ask them how to express the target algebraically in terms of the starting number. Conjecture that if the start number is x, then the target is 10 - x.

- Next have students complete the chains in part B with numbers of their choosing. Have them pair up and conjecture an expression for the target. They should come up with 11 - x. Similarly, have them complete parts C and D. (The resulting expressions are 12 - x and 13 - x, respectively.)

- Point out to students that in parts A - D, only the third sum varied. Have them notice the effect of this on the target expression (as the third sum increases by 1, so does the number in the expression.) Explain that in succeeding problems the other sums will be varied.

- At this point, a spreadsheet could be used to speed up the process for the remaining chains. Depending on the students' capabilities, students can create the spreadsheet themselves, or you can prepare it ahead of time. If students create it, the amount of guidance you give them will of course depend on their amount of experience with spreadsheets.

- The spreadsheet could be set up as follows:

	A	B	C	D	E	F	G
1	Start						Target
2			=B3-A2		=D3-C2		=F3-E2
3		7		11		15	

- The start number is to be typed by the student into cell A2. Of course, the contents of cells B3, D3, and F3 would be changed for each different part of Worksheet 1, and the spreadsheet would be extended to the right for the 4-link and 5-link chains on Worksheet 2. Have students work in pairs to do parts E - G, choosing their own starting numbers, and writing expressions for the target. They should get the following expressions: E = 11 - x; F = 9 - x; and G = 8 - x.

Ask them how the second sum affects the target expression. They should notice that as the second sum increases, the number in the expression decreases by the same amount.

- Have students similarly complete parts H - J. They should get the following expressions: H = 9 - x; I = 11 - x; and J = 12 - x. They should notice that as the first sum increases, the number in the expression increases by the same amount.

- Challenge students to find a general expression for the target in terms of the starting number and the sums a, b, and c (from left to right). Have them look over the work they have done so far, and try to find a relationship between the sums and the number in the target expression. Some students will discover that the number comes from the third sum minus the second sum plus the first sum. In other words, the target is c - b + a - x.

- Have students prove that this expression is correct by completing each square in part K. In the squares they should write (from left to right) x, a - x, b - (a - x) which simplifies to b - a + x, and c - (b - a + x) which simplifies to c - b + a - x.

- Have students work in pairs to complete the 4-link chains on <u>Worksheet 2</u>. (If they are using spreadsheets, they will need to extend them to the right.) They should choose their own starting numbers, and use any numbers for the four sums in the third and fourth chains. Have them conjecture and then prove expressions for the target. Have them generalize using a, b, c, and d for the sums. The expression for the first two chains is 9 + x, and the general expression is d - c + b - a + x. This is proved by placing the following expressions in the five squares:
  x
  a - x
  b - (a - x) = b - a + x
  c - (b - a + x) = c - b + a - x
  d - (c - b + a - x) = d - c + b - a + x

- Have students go on to the rest of <u>Worksheet 2</u>. They should conjecture a formula for the target in the first two 5-link chains, and test it on starting numbers of their choice. (The expression is 13 - x.) Have them conjecture, test, and prove a general expression for the target of a 5-link chain, using the last two chains on the worksheet. (The formula is e - d + c - b + a - x, where the sums are a, b, c, d, and e.)

Assessment:

Have students write down a proposed expression for the target of a 6-link chain. They should write up a demonstration that it works for specific start numbers and sums. Then they should prove that the formula is correct. Their work can be scored on the following scale:

4	Work is correct and complete.
3	Work is almost correct and complete; there is a minor error or a minor detail is omitted.
2	Work shows some understanding, but notable gaps or errors are present.
1	Some work is correct, but there is only minimal understanding, and little or no chain of reasoning.
0	Work is wrong or meaningless. There is no evidence of understanding.

Extension/Follow Up:

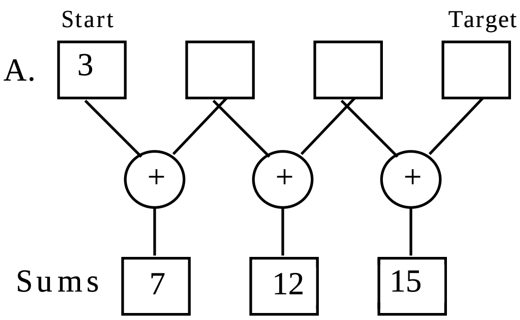
Students can study “arithmetic rings,” in which the chains loop around so there is no starting or target number. In this case, all squares are left blank, and students must find the correct numbers to satisfy all sums simultaneously. It turns out that if the number of links (and therefore the number of squares to be completed) is odd, then there is always a unique solution. Students can find an expression for one number in that solution in terms of the sums. Then, the other numbers can easily be found. If the number of links is even, then there is only a solution if the sums satisfy a particular equation. Students can be challenged to find that equation. In this case, there is an infinite number of solutions; any number can be placed in a square; and the others determined from that number. Another extension is to do a similar investigation when other operations are used instead of addition.

Author:

Geoffrey D. Birky
Greenwood Mennonite School
Private
Sussex County, Delaware

Worksheet 1

A.



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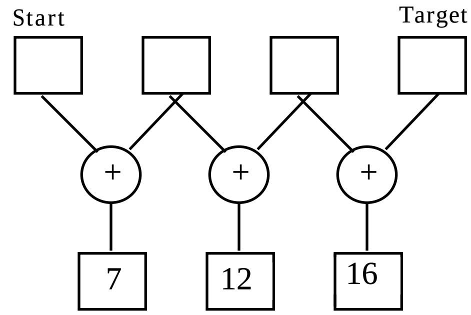

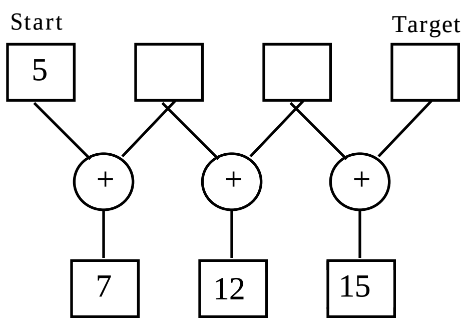

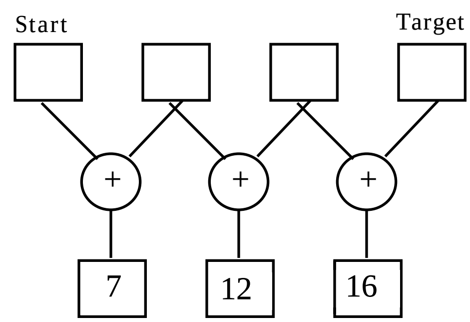

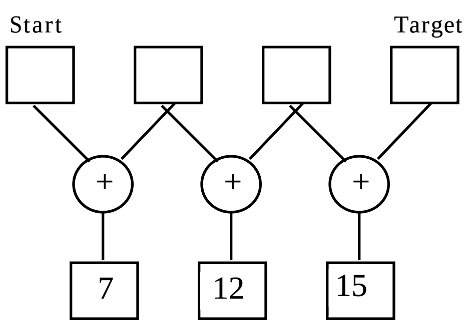

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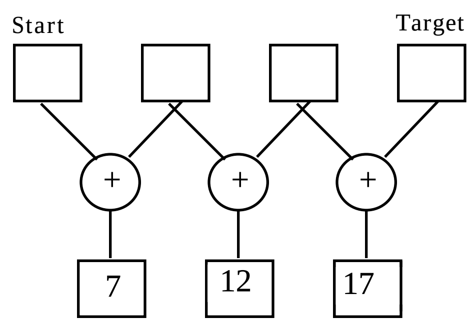

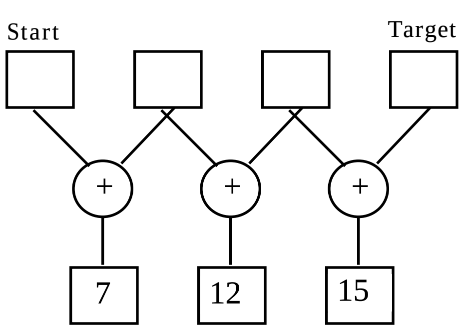

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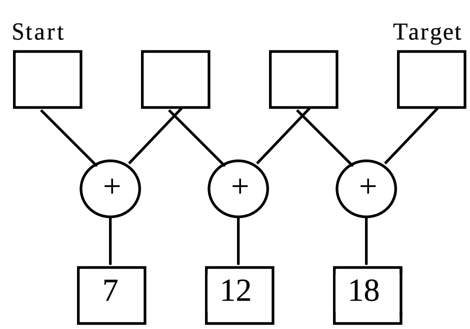
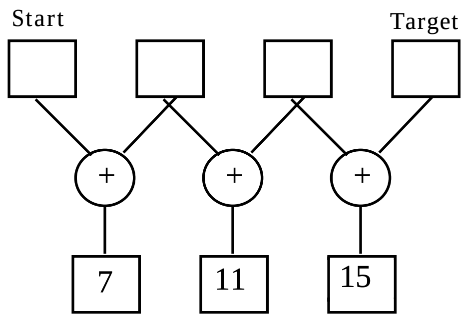
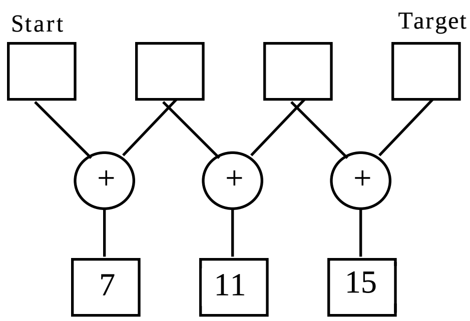


Worksheet 1 (continued)

E.





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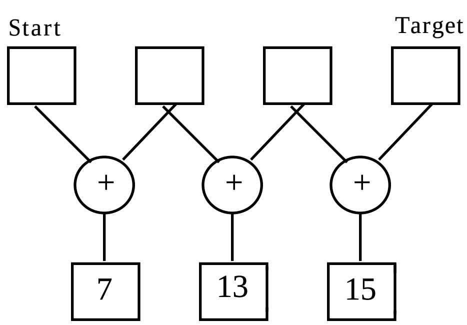

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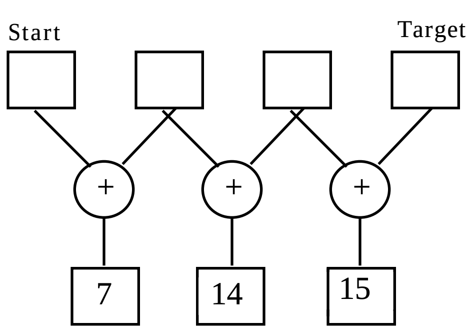

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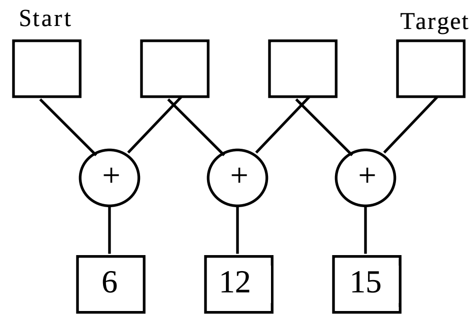

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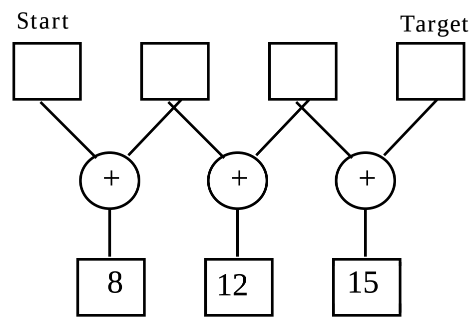

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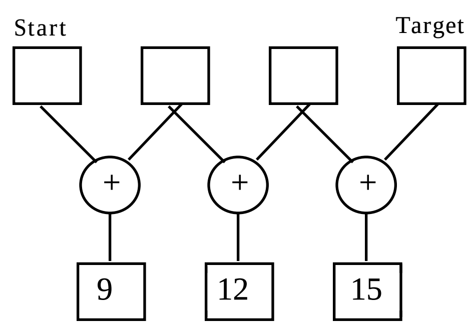

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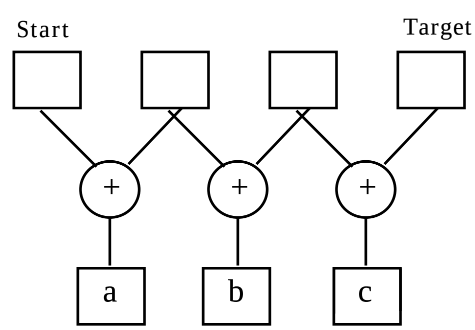
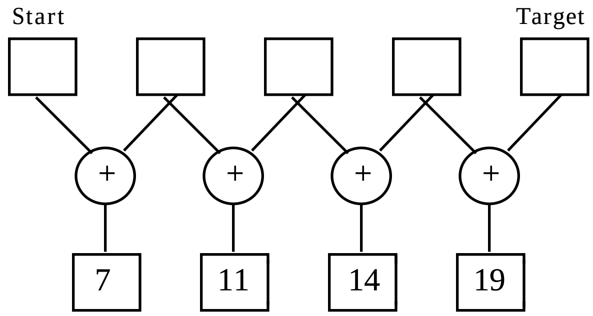


Worksheet 2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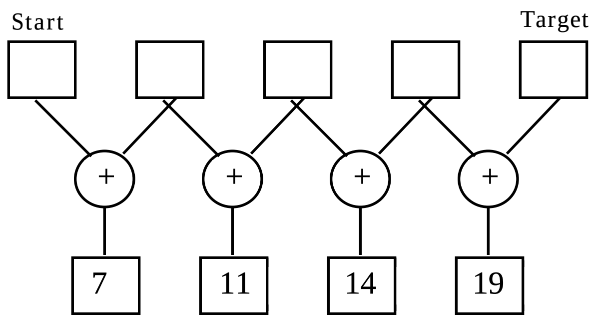
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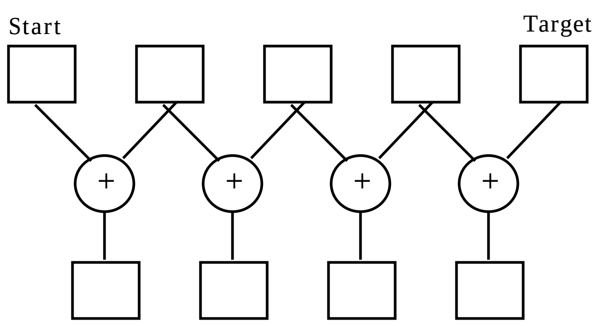
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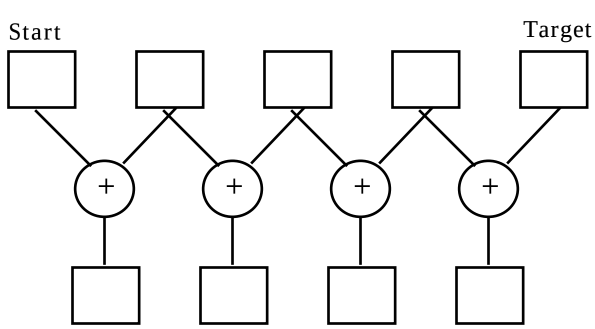
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Worksheet 2 (continued)